Institute of Theoretical Physics
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TENTAMEN GENERAL RELATIVITY

thursday, 6-11–1997, 9.00-12.00, room 5113.0201

Indicate at the first page clearly your name, address, date of birth, year of arrival and at every other page your name.

Question 1

The Robertson-Walker metric for $k = 1$ can be written in the form (we take $c = 1$)

$$ds^2 = dt^2 - R(t)^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\}\,. \tag{1}$$

(1.1) Show that for geodesics with $\dot{\theta} = \dot{\phi} = 0$ the quantity

$$R(t)^2\dot{\chi} \tag{2}$$

is constant. The dot indicates differentiation with respect to an affine parameter.

(1.2) Show that for lightlike geodesics with $\dot{\theta} = \dot{\phi} = 0$

$$R(t)\frac{d\chi}{dt} = \pm 1\,. \tag{3}$$

We consider a closed Friedmann universe. The function $R(t)$ corresponding to such a universe satisfies the differential equation

$$\left(\frac{dR}{dt}\right)^2 + 1 = \frac{A^2}{R}\,, \tag{4}$$

where A is a constant. We impose the boundary condition that $R = 0$ for $t = 0$.

(1.3) Show that the solution of the differential equation (4) is given by the equations

$$\begin{aligned} R &= \tfrac{1}{2}A^2(1-\cos\psi)\,, \\ t &= \tfrac{1}{2}A^2(\psi-\sin\psi)\,, \end{aligned} \tag{5}$$

where ψ is a parameter. Give the graph of the function $R(t)$.

(1.4) Derive an expression for the time $t_0 > 0$, in terms of the constant A, at which for the first time after $t = 0$ the radius R becomes again zero. i.e. $R(t_0) = 0$.

(1.5) At a time $t_1 << A^2$ a foton is emitted from a point P and this foton starts following a geodesic in the plane with $\theta = \phi = \pi/2$. The point P has constant spacelike coordinates with coordinates $\chi = 0$ and $\theta = \phi = \pi/2$. Calculate the time it takes for the foton to return to P.

Question 2

Two observers, A and B, find themselves in free fall in orbits of constant r in the Schwarzschild metric (we take $c = 1$)

$$ds^2 = (1-\frac{2m}{r})dt^2 - (1-\frac{2m}{r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{6}$$

The orbits are in the plane $\theta = \pi/2$ and have a radius $r_A = 4m$, $r_B = 4^{4/3}m$. At the time $t = 0$ the observers A and B both go through the point with $\phi = 0$.

For circular, timelike geodesics with radius r we have

$$(\dot{t})^2 = \frac{r}{r-3m} = (\dot{\phi})^2 r^3/m\,, \tag{7}$$

where the dot indicates differentiation with respect to the eigentime.

(2.1) Calculate the coordinate time Δt_A that observer A needs for one revolution.

(2.2) How much time $\Delta\tau_A$ does observer A need for one revolution according to his own watch?

(2.3) The clock of observer B is lighted and can be read from a distance by observer A. What time difference $(\Delta\tau_B)'$ does A see at the watch of B between two successive passages of B through the point with $\phi = 0$? How much time $(\Delta\tau_A)'$ has evolved in the meantime according to his own watch? Which time difference is larger, $(\Delta\tau_A)'$ or $(\Delta\tau_B)'$? Hint: Use your calculator to estimate the numbers involved.

(2.4) Observer A now uses his rocket motors to stop at the point with coordinates $r = 4m, \theta = \pi/2, \phi = 0$. He repeats the observations of question (2.3). What are now the results of these observations?

Question 3

The Riemann tensor in N spacetime dimensions has components $R^d{}_{abc}$. We define $R_{abcd} = g_{ae}R^e{}_{bcd}$. The Riemann tensor satisfies the identities

$$\begin{aligned} R_{abcd} &= -R_{bacd} = -R_{abdc}\,, && (8)\\ R_{abcd} &+ R_{adbc} + R_{acdb} = 0\,. && (9) \end{aligned}$$

(3.1) Show, by using equations (8) and (9), that

$$R_{abcd} = R_{cdab}\,. \tag{10}$$

From now on we assume that the number of spacetime dimensions N is equal to 3.

(3.2) Show that for $N = 3$ the Riemann tensor can be expressed in terms of the Ricci tensor as follows:

$$R^{ab}{}_{cd} = -\frac{1}{g}\epsilon^{abe}\epsilon_{cdf}(R^f{}_e - \tfrac{1}{2}\delta^f{}_e R)\,, \tag{11}$$

where

$$\begin{aligned} R^{ab}{}_{cd} &= g^{be}R^a{}_{ecd}\,, && (12)\\ g &= \det\,(g_{ab})\,. && (13) \end{aligned}$$

and ϵ^{abc} is the completely anti-symmetric Levi-Civita symbol in 3 dimensions ($\epsilon^{012} = 1$).

Hint: Use the identities (without proof)

$$\epsilon^{acd}\epsilon_{bef}R^{ef}{}_{cd} = -4g(R^a{}_b - \tfrac{1}{2}\delta^a{}_b R)\,, \tag{14}$$

$$\epsilon^{abe}\epsilon_{cde} = g(\delta^a{}_c\delta^b{}_d - \delta^b{}_c\delta^a{}_d)\,. \tag{15}$$

(3.3) Show that the result of question (3.3) implies that every solution of the Einstein equation with $T_{ab} = 0$ describes a flat space.

(3.4) Take $T_{ab} = \Lambda g_{ab}$, with Λ is constant. Show that every solution of the Einstein equation corresponds to a maximally symmetric space.